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Theory of the Eulerian tail in the spectra of atmospheric and oceanic internal gravity waves

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Observed atmospheric and oceanic internal wave spectra, when analysed in an Eulerian frame of reference, exhibit a large-wavenumber 'tail'. In one-dimensional vertical-wavenumber (k_3) spectra, it is typically proportional to $|k_3|^{-3}$.

In 1989, K. R. Allen and R. I. Joseph showed that a large-wavenumber tail was to be anticipated as a consequence of Eulerian nonlinearity, and they derived relations for the coefficients of both horizontal and vertical spectra of the form $|k|^{-3}$. The coefficients were obtained only for the wave-induced vertical-displacement spectra, and only for an input spectrum having a certain 'canonical' frequency variation derived on other grounds.

The present work builds on that of Allen & Joseph. It is more general in some respects, more limited in others. It provides a more transparent form of analysis, it treats a broad class of wave variables, and it does so for input (Lagrangian) spectra that can be chosen by the user, free from any constraint to canonical or other restricted forms. It provides relations whereby the full Eulerian spectrum may be determined numerically, once the input spectrum has been chosen, and it provides analytic forms applicable at large wavenumbers for horizontally isotropic spectra. The derived one-dimensional vertical-wavenumber spectra are discussed in relation to observations.

Certain shortcomings in the development, both as given by Allen & Joseph and as found here, are identified and discussed.

1. Introduction

Observations in the middle atmosphere and in the oceans reveal fluctuations thought to be induced by internal gravity waves. Portions of their Eulerian spectra often reproduce themselves with little change from one occasion to another, as was recognized first by Garrett & Munk (1972, 1975) for the oceans and then by VanZandt (1982) for the atmosphere. Perhaps the most notorious example of this behaviour is given by the vertical-wavenumber (k_3) spectrum of small-scale horizontal winds in the middle atmosphere. Its form is typically close to $|k_3|^{-3}$, with a coefficient that, after division by the stability N^2, varies but little regardless not only of circumstances but even of height (e.g. Dewan *et al.* 1984; Tsuda *et al.* 1989) despite the growth with height that might be expected to result from the diminution of gas density. The corresponding oceanic multiple of stability is virtually the same, which fact is suggestive of a common mechanism of control. There are many who believe that such spectra, whether in the atmosphere or in the ocean, represent the effects of the Eulerian advective nonlinearity acting on the wave system.

As was pointed out by Allen & Joseph (1989; hereinafter AJ89), no such nonlinearity arises in a Lagrangian formulation of the problem. The process was therefore amenable to examination by use of a quasi-linearized Lagrangian formulation accompanied by a transition from the Lagrangian to the Eulerian spectral description.

AJ89 provided a method of achieving the transition. They applied it to the spectra of vertical displacements induced by a certain 'canonical' spectrum that was thought to be appropriate to oceanic internal waves. Chunchuzov (1996) adapted and extended the method for vertical displacements in the atmosphere. Both analyses employed the slightly arcane correlation techniques of spectral analysis, and the generality of the conclusions was restricted by the form assumed for the input Lagrangian spectrum.

A new form of adaptation is presented here. It is conducted in what will be, to many, a far more transparent manner. It permits evaluation for a wide variety of Lagrangian spectra and of wave parameters, and it thereby reveals both the degree of generality of the conclusions and their variability from one spectrum or observed fluctuation to another.

The paper is organized as follows. In § 2, we specify the form of the ensemble of Lagrangian waves that is assumed. It requires, in part, that the Lagrangian waves be legitimately linearized (the conditions for which are outlined in Appendix A). In § 3, we transform the associated waveform from Lagrangian into Eulerian coordinates. In § 4, we obtain formally the Fourier transform of the Eulerian waveform and indicate the origins of the large-wavenumber tail that it implies. In § 5, we convert this formally into a four-dimensional power spectral density, given by (5.2), whose evaluation requires the determination of a particular space–time average to produce a correlation function. In §§ 6 and 7, the averaging is conducted in a transparent fashion for the assumed case of a large number of uncorrelated waves. The correlation functions that then result are given in § 8, requiring only insertion in (5.2) and numerical integration in order to produce the Eulerian spectra that correspond to the input Lagrangian ensemble.

Subsequent sections are concerned only with the spectra at large Eulerian wave vectors $\boldsymbol{k}$, to be identified as the Eulerian spectral tail. Section 9 obtains the general form of the functions required for the production of the lead term of an asymptotic expansion of the four-dimensional spectrum, in the fashion of AJ89 (but altered from AJ89 as in Appendices B and C). It is amplified by Appendix D, where detailed expressions are given for integrands required for the lead term of the three-dimensional wavenumber spectrum for each of displacement and velocity. The method of integration of these lead terms to produce spectral densities, as required in (5.2), is indicated in § 10, with explicit expressions for the integrals being given in the related Appendix E. Numerical integration is required to convert these into one-dimensional spectra, but § 11 provides a crude analytic assessment of the vertical-wavenumber spectra of displacement and velocity (with the aid of Appendix F).

The important results and observational implications of the analysis, including universal $|k_3|^{-3}$ spectra, are discussed in § 12.

Throughout the development, repeated comparison is made with AJ89. This serves to underline both the debt owed to that work and the points of contrast to it. The latter are significant not only in their own right but also for assessment of the validity of the new results where these differ seriously, as in places they do, from those of AJ89.

2. The Lagrangian input spectra

We begin with the assumption of an input spectrum consisting of a multitude of gravity waves, each having such small amplitude that a quasi-linear Lagrangian description of the total oscillation applies (see Appendix A). At an early stage, we assume incompressibility as a simplification, but we outline the consequences of compressibility in Appendix G. For simplicity of discussion only, we ignore Coriolis effects; they could be included (see Appendix G). We also ignore dissipation, though this is a more fundamental omission that requires some discussion (given in outline in § 12). The standard growth of amplitude with height that occurs in the atmospheric case is also ignored in the mathematical development, with justification (insofar as local application is concerned) at large vertical wavenumbers. Finally, we ignore steady background flow, because of the complications it imposes on a Lagrangian analysis if it is shearing, and its irrelevance if it is not.

The Lagrangian coordinate $\boldsymbol{r}$ of a fluid parcel is taken to be the Eulerian coordinate $\boldsymbol{x}$ of the parcel in the absence of waves. The parcel retains that Lagrangian coordinate for all time, t, no matter what wave-induced Eulerian displacement $\boldsymbol{s}$ it may experience.

The input wave system is defined in terms of the Lagrangian wavenumber components $\alpha_{mj}(j = 1, 2, 3;$ 3 is upward) appropriate to each mode m, each being separately identifiable in the Lagrangian-linear approximation. We represent the parcel's displacement $\boldsymbol{s}_m$ induced by mode m as $\boldsymbol{S}_m c\{m\}$, where $\boldsymbol{S}_m$ is a real vector and $c\{m\} \equiv \cos[\alpha_{mj} r_j - \beta_m t - \phi_m]$. (Summation over a repeated j is implied, here and throughout. All three components of $\boldsymbol{S}_m$ share a common phase, as is implied, under present approximations.) The frequency β_m is defined to be positive, being given under present approximations by $\beta_m = N\alpha_{mh}/|\alpha_{m3}|$, where N is the buoyancy frequency and α_{mh} is the magnitude of the horizontal component $\boldsymbol{\alpha}_{mh}$ of $\boldsymbol{\alpha}_m$. Vertical wavenumbers α_{m3} are accordingly negative for upgoing waves, positive for downgoing waves. The opposite prescription is made for the amplitude factor of the vertical component of displacement, S_{m3}: it is positive for upgoing waves and negative for downgoing waves. Absence of Coriolis effects then ensures that the horizontal displacement amplitude vector $\boldsymbol{S}_{mh}$ will be in the direction of $\boldsymbol{\alpha}_{mh}$, and incompressibility ensures that $S_{mh}\alpha_{mh} + S_{m3}\alpha_{m3} = 0$. These or alternative conventions must be adhered to consistently.

The input spectrum consists of some number $\hat{N}$ of modes specified by $S_{m3} = S_{m3}[\boldsymbol{\alpha}_m]$, with $m = 1, 2, \ldots \hat{N}$; $\boldsymbol{\alpha}_m$ is restricted to a fixed finite region of wavenumber space. We anticipate transiting to a continuous spectrum, one in which the number of modes $\hat{N}$ has tended to infinity. In the course of the transition, we require each $\boldsymbol{S}_m$ to be proportional to $\hat{N}^{-1/2}$ in order to maintain the finiteness of energy density and the integrity of Lagrangian linearity.

We wish to evaluate the Eulerian spectrum of some scalar field variable L that, in the Lagrangian frame, can be related to the vertical displacement on a mode-by-mode basis by use of the $S_{m3}c\{m\}$ of that mode and the polarization relations. The contribution from the m mode is given by $L_m c\{m\}$ or, as may be the case, by $L_m s\{m\}$ where $s\{m\}$ is the sine equivalent of $c\{m\}$; or in general by a combination of the two (with the sign of L_m being defined by the polarization relation). These two cases behave somewhat differently as we progress, so the distinction must always be kept in mind.

The analysis is limited to variables L that take the same value in both coordinate systems and that relate to a single parcel of fluid at a time. Displacement and velocity of a parcel are such quantities, whereas velocity shear is not; it would be obtained by differentiating velocity after the transformation of coordinates.

Our objective is to determine the Eulerian power spectral density of L corresponding to any given Lagrangian amplitude spectrum $S_{m3} = S_{m3}[\boldsymbol{\alpha}_m]$ that maintains quasi-linearity in the Lagrangian frame, for large $\hat{N}$.

3. Lagrangian → Eulerian transition

As previously noted, the Lagrangian coordinate $\boldsymbol{r}$ of a fluid parcel is taken to be the Eulerian coordinate $\boldsymbol{x}$ of the parcel in the absence of waves. In the presence of the waves, the parcel experiences vector displacement $\boldsymbol{s}[\boldsymbol{r}, t] = \sum \boldsymbol{s}_m[\boldsymbol{r}, t]$, where the sum extends over all m. Accordingly, the parcel lying at some chosen $\boldsymbol{x}$ at some chosen time t is the parcel whose $\boldsymbol{r}$ is given implicitly by

$$\boldsymbol{x} = \boldsymbol{r} + \boldsymbol{s}[\boldsymbol{r}, t]. \tag{3.1}$$

The required parameter L associated with the parcel has the same value in the two coordinate systems, though for convenient identification we name it E in the Eulerian system: $E[\boldsymbol{x}, t] = L[\boldsymbol{r}, t]$, where $\boldsymbol{x}$ and $\boldsymbol{r}$ are related by (3.1). This fact can be expressed formally with the aid of a three-dimensional delta function:

$$E[\boldsymbol{x}, t] = \int \mathrm{d}^3 r L[\boldsymbol{r}, t]\, J[\boldsymbol{s}]\, \delta[\boldsymbol{x} - \boldsymbol{r} - \boldsymbol{s}], \tag{3.2}$$

in which $J[\boldsymbol{s}]$ is the Jacobian of the coordinate transformation (AJ89). We take $J = 1$ here, in keeping with the assumed incompressibility, but we note the potential relevance of $J \neq 1$ and revisit it in Appendix G. Integrations extend from −infinity to +infinity here and elsewhere unless otherwise indicated.

The mapping expressed by (3.2) is in general intractable. Its potential for spectral studies was, however, recognized and exploited by AJ89, where the delta function proper was replaced by a Fourier transform as

$$\delta[\boldsymbol{x} - \boldsymbol{r} - \boldsymbol{s}] = (2\pi)^{-3} \int \mathrm{d}^3 K\, \mathrm{e}^{\mathrm{i}\boldsymbol{K}\cdot(\boldsymbol{x}-\boldsymbol{r}-\boldsymbol{s})}, \tag{3.3}$$

to produce

$$E[\boldsymbol{x}, t] = (2\pi)^{-3} \int \mathrm{d}^3 r\, \mathrm{d}^3 K\, L[\boldsymbol{r}, t]\, \mathrm{e}^{\mathrm{i}\boldsymbol{K}\cdot(\boldsymbol{x}-\boldsymbol{r}-\boldsymbol{s})}, \tag{3.4}$$

though AJ89 retained $J \neq 1$ and included Coriolis effects ignored here. Our path now parts from, but parallels, that of AJ89.

4. The Eulerian tail

The four-dimensional Fourier transform of E may be found as

$$\hat{E}[\boldsymbol{k}, \omega] = (2\pi)^{-3} \iint \mathrm{d}^3 x\, \mathrm{d}t\, \mathrm{d}^3 r\, \mathrm{d}^3 K\, \mathrm{e}^{-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)} L[\boldsymbol{r}, t]\, \mathrm{e}^{\mathrm{i}\boldsymbol{K}\cdot(\boldsymbol{x}-\boldsymbol{r}-\boldsymbol{s})}. \tag{4.1}$$

Integration over $\boldsymbol{x}$ introduces a delta function in $(\boldsymbol{k} - \boldsymbol{K})$, and integration over $\boldsymbol{K}$ inserts a $\boldsymbol{k}$ where $\boldsymbol{K}$ was previously found. The net result is

$$\hat{E}[\boldsymbol{k}, \omega] = \int \mathrm{d}^3 r\, \mathrm{d}t\, \mathrm{e}^{-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} L[\boldsymbol{r}, t]\, \mathrm{e}^{-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{s})}. \tag{4.2}$$

This form warrants examination before we proceed.

If (4.2) lacked its final exponential factor, it would provide identically the Lagrangian Fourier transform. If $\boldsymbol{k} \cdot \boldsymbol{s}$ is sufficiently small, then, the Eulerian transform must differ negligibly from the Lagrangian; the two will have virtually identical power spectra. However, the 'sufficiently small' criterion can be breached either by considering sufficiently large wave amplitudes at fixed $\boldsymbol{k}$, or by considering sufficiently large $\boldsymbol{k}$ even with small wave amplitudes. In the latter case, even for the smallest

Lagrangian wave amplitudes that we wish to consider, the Eulerian spectrum must have a large-wavenumber 'tail' that we can find (formally) merely by looking for it. Its existence does not depend on the occurrence of large Lagrangian wave amplitudes, and the two types of spectra are intrinsically different in nature.

With respect to the vertical-wavenumber spectra, the transition must set in at $|k_3|$ no greater than the inverse of the r.m.s. vertical displacement $\langle s_3 \rangle$. For gravity waves, this inverse is approximately N divided by the r.m.s. horizontal perturbation velocity. Equivalently, then, the transition occurs at horizontal phase speeds ($\omega/k_h \simeq |N/k_3|$) comparable to the r.m.s. horizontal perturbation velocity, as is expected also from the form of the advective nonlinearity of the Eulerian equations. From this transition point on, if not before, the Eulerian spectrum contains components not duplicated in the Lagrangian spectrum – components that are spread (in wavenumber–frequency space) from the individual modes of the Lagrangian spectrum.

It should be noted expressly that the occurrence of the large-wavenumber Eulerian tail has nothing whatever to do with any physical process; it is purely a kinematic consequence of what might be termed looking at the waves in the wrong coordinate system. If one insists on defining 'waves' according to their Eulerian linear description, then one is forced to admit to the existence of 'nonlinear wave–wave interactions' in order to produce the tail, but these are mere mathematical artefacts and have no physical import. The pseudowaves that clutter the tail are not waves at all; they do not obey dispersion or polarization relations appropriate to true waves, nor do they propagate freely. They simply mark deformations imposed on waves by other waves, locally enhancing their shears, as has been illustrated by Eckermann (1999). This is true even though the underlying waves may still be identified as such in the Lagrangian description, complete with their dispersion and polarization relations, as was established by AJ89.

It may be noted in passing that there is no similarly generated Eulerian frequency tail, because the Lagrangian → Eulerian transformation involves a spatial mapping only. Nevertheless, Eulerian frequencies must be distinguished from input Lagrangian frequencies in the tail regime, they being something of an artifical construct as well. They may, for example, include frequencies exceeding N (AJ89). Within an Eulerian framework, they may be viewed heuristically as having been produced by a Doppler spreading caused by the velocity fluctuations induced by the wave system itself (Hines 1991*b*).

5. The four-dimensional Eulerian power spectral density (PSD)

Resuming our formal development, we next obtain the complex conjugate of $\hat{E}$, namely

$$\hat{E}^*[\boldsymbol{k}, \omega] = \int \mathrm{d}^3 r' \, \mathrm{d}t' \, \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}' - \omega t')} L'[\boldsymbol{r}', t'] \, \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{s}')}, \tag{5.1}$$

in an obvious notation, employing a distinct set of Lagrangian coordinates $(\boldsymbol{r}', t')$ for the purpose. We form the product of (4.2) with (5.1), change $\boldsymbol{r}'$ and t' into $\boldsymbol{r} + \boldsymbol{R}$ and $t + T$, and conduct the integrations over $\boldsymbol{r}'$ and t' via the new variables $\boldsymbol{R}$ and T assuming statistical stationarity. As defined for an arbitrarily large volume of space–time V_4, the Eulerian power spectral density (with radian units, which are to be employed here) corresponding to L is then given by $[2\pi]^{-4} V_4^{-1}$ times the result:

$$PSD[E] = [2\pi]^{-4} V_4^{-1} \hat{E} \hat{E}^* = [2\pi]^{-4} \int \mathrm{d}^3 R \, \mathrm{d}T \, \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{R} - \omega T)} A, \tag{5.2}$$

where

$$A \equiv A[\boldsymbol{R}, T] \equiv V_4^{-1} \int_{V_4} \mathrm{d}^3 r \, \mathrm{d}t \, LL' \, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{s}'-\boldsymbol{s})}, \tag{5.3}$$

with $L' \equiv L[\boldsymbol{r}+\boldsymbol{R},\, t+T]$ and likewise $\boldsymbol{s}'$. This A, which gives the space–time average of $LL' \exp[\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{s}'-\boldsymbol{s})]$ for any chosen $\boldsymbol{R}$, T, is equivalent to the 'expectation value' of the same quantity for the same $\boldsymbol{R}$, T, defined by AJ89 as an integral over a displacement-momentum phase space (presumably under an assumption of ergodicity). We will evaluate it, instead, by contemplating averages over the phases of all $c\{m\}$ and $s\{m\}$ – a much more transparent process. For the purpose, we will need also displacements of phase $\delta m \equiv \alpha_{mj} R_j - \beta_m T$ and then corresponding cosine and sine factors, $c\{\delta m\}$ and $s\{\delta m\}$, with which to express L' and $\boldsymbol{s}'$ as modified forms of L and $\boldsymbol{s}$.

If it were not for the exponential factor in (5.3), the averaging process would be straightforward. L and L' would be expressed as sums over all modes, then multiplied together. Upon averaging, product terms of the type $L_m L'_n$ would vanish for $m \neq n$ because of mutual phase incoherence. Terms of the type $L_m L'_m$ would produce $c\{m\}$ and $(c\{m\}c\{\delta m\} - s\{m\}s\{\delta m\})$ factors for in-phase variables, $s\{m\}$ and $(s\{m\}c\{\delta m\} + c\{m\}s\{\delta m\})$ factors for phase-quadrature variables. Averaging would then eliminate cross-products of the form $c\{m\}s\{m\}$ and yield terms of the form $\frac{1}{2}L_m^2 c\{\delta m\}$ both for in-phase and for phase-quadrature variables. The sum over m of all such averages, if inserted in (5.2) and integrated there, would give the Lagrangian power spectral density of L corresponding to the chosen set of $S_{m3} = S_{m3}[\boldsymbol{\alpha}_m]$. (Note that, whereas the input spectrum was defined to contain only positive frequencies for later convenience, this output spectrum contains both positive and negative frequencies as did that of AJ89. It also contains both positive and negative wavenumbers, even if all Lagrangian waves are either upgoing or downgoing.)

For the Eulerian power spectrum, the exponential in (5.3) cannot be ignored except at small $|k_j|$ – typically, at $|k_j|$ small in comparison with the inverse of the j component of the corresponding r.m.s. displacement $\langle s_j \rangle$, for all j (= 1, 2 and 3). At such k_j, it would reproduce the Lagrangian spectrum. However, if any $|k_j|$ exceeds this limit, as it will in the Eulerian tail, the exponential factor in (5.3) must be taken into account. Correlations will occur between space–time variations in LL' and those in the exponential factor, and they include correlations even between modes with $m \neq n$. Our immediate task is to determine these correlations.

6. Evaluation of the correlations

The nature of the correlation varies as between L_m being in phase or in phase quadrature with respect to S_{m3}. We examine first the in-phase type of L and concentrate for the moment on two specific wave modes, designated m in the L sum and n in the L' sum, respectively.

If $m = n$, there is only a single factor in $\exp[\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{s}'-\boldsymbol{s})]$ with which a correlation can be found, namely $\exp[\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{s}'_m - \boldsymbol{s}_m)]$. We extract this factor from the exponential for separate treatment, expanding it in a Taylor series as $1 + \mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{s}'_m - \boldsymbol{s}_m) + \cdots = 1 + \mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{S}_m)(c\{m\}[c\{\delta m\} - 1] - s\{m\}s\{\delta m\}) + \cdots$. (With a multitude of waves and finite energy, $|\boldsymbol{S}_m|$ is small and the expansion is justified not only for small but even for large k; although $\boldsymbol{k}\cdot\boldsymbol{s}$ may not be small, each individual $\boldsymbol{k}\cdot\boldsymbol{S}_m$ will be, the more so as the number of modes $\hat{N}$ tends to infinity.) The product term $L_m^2 c^2\{m\}c\{\delta m\}$ combines with the first term of the expansion to produce $\frac{1}{2}L_m^2 c\{\delta m\}$ on average, just as before.

Upon summation, it yields

$$M \equiv \sum [L_m^2 c\{\delta m\}]/2 \tag{6.1}$$

just as before, but now the remainder of the original $\exp[\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{s}'-\boldsymbol{s})]$ factor is left over for subsequent incorporation. The remainder of the present expansion is likewise left over, but it tends to 0 as $\hat{N}$ tends to infinity and will not be discussed further.

When $m \neq n$, two exponential factors must be extracted for this separate treatment, those in m and n. Upon expansion, together they yield $1 + \mathrm{i}[\boldsymbol{k}\cdot(\boldsymbol{s}'_m - \boldsymbol{s}_m + \boldsymbol{s}'_n - \boldsymbol{s}_n)] - \frac{1}{2}[..]^2 + \ldots$, where $[..]$ stands for the preceding bracketed expression. The multiplier LL' contributes a term with coefficient $L_m L_n c\{m\}(c\{n\}c\{\delta n\} - s\{n\}s\{\delta n\})$ for correlation with this expansion. The first two terms of the expansion, expressed as 1 plus a sum of multiples of $c\{m\}$ and its partners, yield zero correlation since all products contain at least one of $c\{m\}$, $s\{m\}$, $c\{n\}$ and $s\{n\}$ to an odd power. The same is true of the self-product terms of $[..]^2$ – those containing $(\boldsymbol{k}\cdot\boldsymbol{S}_m)^2$ and $(\boldsymbol{k}\cdot\boldsymbol{S}_n)^2$ as multipliers – but not of the cross-product term $2(\boldsymbol{k}\cdot\boldsymbol{S}_m)(\boldsymbol{k}\cdot\boldsymbol{S}_n)(c\{m\}[c\{\delta m\} - 1] - s\{m\}s\{\delta m\})(c\{n\}[c\{\delta n\} - 1] - s\{n\}s\{\delta n\})$. In conjunction with the trigonometric factors of LL', it produces terms such as $c^2\{m\}[c\{\delta m\} - 1]c^2\{n\}c\{\delta n\}[c\{\delta n\} - 1]$, whose average value is $\frac{1}{4}[c\{\delta m\} - 1]c\{\delta n\}[c\{\delta n\} - 1]$, and so on. All such averages must be combined.

There is a mode m member of L' as well, of course, and it contributes similarly in combination with mode n in L. After allowing for that, we may keep m fixed and add over all n to find the net contribution of mode m to LL' as correlated with the exponential factor. Then we may add over all m to find the total of all such contributions, though each pair will then have been counted twice and the sum must be divided by 2. The net result is

$$P \equiv \left[\sum L_m(\boldsymbol{k}\cdot\boldsymbol{S}_m)(1 - c\{\delta m\})/2\right]^2. \tag{6.2}$$

As given, P contains self-products of modes ($m = n$). These should in principle be subtracted, since self-products behave differently and have already been accounted for; but as $\hat{N}$ tends to infinity, their contribution to P tends to 0, so the subtraction process is ignored. Higher-order terms in the expansions of the exponentials could be retained, but their sum too tends to 0 as $\hat{N}$ tends to infinity, so they too are ignored.

The result of these processes is to yield $(M + P)$ as the correlated factor that emerges from (5.3). Of course, each of the product terms should have been multiplied by the contributions to $\exp[\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{s}' - \boldsymbol{s})]$ of the remaining modes at each point of space–time in turn. However, since there is no correlation between those modes and the m and n modes, their averaging process can be conducted separately. That will be done shortly, after we deal with the correlations associated with phase-quadrature variables.

For them, the steps that led to M for in-phase variables may be repeated and M is again found, now via an averaging of $s^2\{m\}$ rather than $c^2\{m\}$. But the steps that led to P now yield, in place of it,

$$-Q \equiv -\left[\sum L_m(\boldsymbol{k}\cdot\boldsymbol{S}_m)s\{\delta m\}/2\right]^2. \tag{6.3}$$

(The negative sign should be noted.) This parameter, like P, is valid at all $\boldsymbol{k}$ as $\hat{N}$ tends to infinity. It joins with M to produce $(M - Q)$ for subsequent union with the remainder of the $\exp[\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{s}' - \boldsymbol{s})]$ factor. We turn next to the averaging of that factor.

7. Averaging of $\exp[\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{s}'-\boldsymbol{s})]$

As noted, each of the contributing m, m and m, n terms should have been multiplied by the product of the $\exp[\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{s}_q'-\boldsymbol{s}_q)]$ factors of all non-correlating modes q ($\neq m, \neq n$) in the course of averaging over space–time. However, also as noted, that product may be separately averaged because there is no correlation to be taken into account. (We ignore here putative resonant interactions. They find no place in a linearized Lagrangian formulation, however much they may appear to exist when the Eulerian variations are attributed to wave–wave interactions. Their role might, perhaps, be a matter for future study.) We may include even the exponential factors of the m and n modes in the averaging process, since they make a negligible contribution to the mean, and so we seek the mean of the entire exponential factor $\exp[\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{s}'-\boldsymbol{s})] \equiv \exp[\mathrm{i}g]$, say.

The parameter g is the sum of a multitude of independent sinusoids lacking phase coherence with one another. The probability of its having a value within some range dg about some chosen value g is therefore given, by the central limit theorem, as

$$p[g]\,\mathrm{d}g = (2\pi G)^{-1/2}\exp[-g^2/2G]\,\mathrm{d}g, \tag{7.1}$$

where G is the mean-square value (the variance) of g. The latter can be determined by expanding g as the sum of individual modes once again, squaring, eliminating cross-product terms via phase incoherence, and averaging $\cos^2$ and $\sin^2$ factors of the self-product terms as $\frac{1}{2}$. This process yields, after division by 2 for later convenience,

$$G/2 = \sum(\boldsymbol{k}\cdot\boldsymbol{S}_m)^2(1-c\{\delta m\})/2. \tag{7.2}$$

The mean value of the exponential factor in (5.3), taken by itself, is then found to be

$$\int \mathrm{d}g(2\pi G)^{-1/2}\exp[-g^2/2G]\exp[\mathrm{i}g] = \exp[-G/2]. \tag{7.3}$$

Here as later, use has been made of a more general integral cited by AJ89:

$$\int \mathrm{d}w\, w^n \exp[-a^2w^2 + \mathrm{i}bw] = \mathrm{i}^n\pi^{1/2}2^{-n}a^{-(n+1)}\exp[-b^2/4a^2]H_n[b/2a], \tag{7.4}$$

where a is the positive root of a^2 and $H_n[z]$ is a Hermite polynomial given by $(-1)^n\exp[z^2]\,\mathrm{d}^n(\exp[-z^2])/\mathrm{d}z^n$. ($H_0 = 1$, $H_1 = 2z$, $H_2 = 4z^2 - 2$, etc.) The factor $\exp[-G/2]$, though obtained here in a different way, corresponds exactly to the exponential factor in (3.12) of AJ89.

8. The four-dimensional Eulerian spectrum

The final result for in-phase L is, then,

$$A = A_P \equiv (M+P)\exp[-G/2], \tag{8.1}$$

whereas that for phase-quadrature L is

$$A = A_Q \equiv (M-Q)\exp[-G/2]. \tag{8.2}$$

These forms differ from the corresponding Lagrangian relations by having non-zero P, Q and G. They are designed for insertion into (5.2) and for subsequent integration over $\boldsymbol{R}$ and T there, weighted according to the requirements of (5.2). This process, when completed, yields the Eulerian spectrum corresponding to the input Lagrangian spectrum, for the assumed case $J = 1$ and $\hat{N} \rightarrow$ infinity.

Unfortunately, the integrations required in (5.2) cannot be completed in closed form. In general, the Eulerian spectral results can become available only via numerical methods. However, by inspection we can confirm that at small wavenumbers, such as to make P, Q and G negligibly small, the Eulerian spectrum will differ negligibly from the Lagrangian.

Our interest now turns to the opposite extreme of large wavenumbers $k(\equiv |\boldsymbol{k}|)$, such that at least one component of $\boldsymbol{k}$, multiplied by the corresponding r.m.s. component of displacement, yields a result > 1.

9. Expansions for integration over R_j and T

As was established by AJ89, it is possible to carry the analytic process further if attention is confined to the asymptotic behaviour at sufficiently large $\boldsymbol{k}$. We now proceed along that path, but with use of the present (8.1) and (8.2) rather than the corresponding form $k_1^{-2}M_{33}$ found in (3.11) of AJ89. They are not the same, even for the vertical displacement treated in AJ89, and the distinction becomes important at large wavenumbers. AJ89 is believed to be incomplete on this point, for reasons given in Appendix B concerning certain endpoint contributions.

With attention confined to large wavenumbers, AJ89 argued that the $\exp[-G/2]$ factor acts as a sort of cutoff factor: as k is made progressively larger, it progressively restricts the range of values of $|R_j|$ and $|T|$ from which significant contributions to the integrals in (5.2) may be made. This restriction permitted an expansion of various functions in Taylor series over the restricted range. The lead terms in G itself were quadratic in R_j and T, and these terms alone were retained there. They provided $\exp[-G/2]$ factors having the form $\exp[-k^2w^2]$, identical to the form $\exp[-a^2w^2]$ on the left-hand side in (7.4). The exact integrals given in (7.4), when employed for the evaluation of (5.2), then led to expansion of the asymptotic spectrum in terms proportional to k^{-q}, with $q = 5, 7, \ldots$ in three-dimensional spectra. After integration to produce one-dimensional spectra, the sequence became $q = 3, 5, \ldots$. In both cases, only the coefficient of the lowest-order term was derived, it being the only relevant term in the asymptotic limit of large wavenumber. Here, the q sequence derives from the n sequence of (7.4), with $n = 2, 4, \ldots$ as found in the expansion of coefficients that correspond to the present $M + P$. The use of the approximate G to produce exact integrals in this fashion will be termed the AJ89 protocol.

With the more complete relations obtained here, implementation of the AJ89 protocol would lead to $q = 3, 5, \ldots$ in three-dimensional spectra, $q = 1, 3, \ldots$ in one-dimensional spectra, the new lead terms being associated with the constant (i.e. $n = 0$) component of M in (6.1), namely $\sum L_m^2/2$. The lead term in each sequence is both mathematically and physically unacceptable, since it implies (after integration over wavenumber) infinite Eulerian variances in association with finite Lagrangian variances. As is shown in Appendix C, implementation of the AJ89 protocol in application to the case $n = 0$ introduces an error proportional to $|k|^{-3}$ in three-dimensional spectra, $|k|^{-1}$ in one-dimensional spectra. When this error is removed, as it can be unambiguously for in-phase variables, it yields the expected sequences $q = 5, 7, \ldots$ in three-dimensional spectra and $3, 5, \ldots$ in one-dimensional spectra. We therefore simply ignore the erroneous lead term that the AJ89 protocol produces in the analysis here, and present only the next term. (The revision is less readily justified in the case of phase-quadrature variables if the input spectrum lacks vertical symmetry, but must be equally valid; see

Appendix C.) In this way we come into agreement with AJ89 as to the form of the lead term, but the coefficient will differ because of the endpoint portion overlooked by AJ89.

With the AJ89 protocol now known to produce error when applied to the case $n = 0$, some suspicion of error in the coefficients of the higher-order terms may arise. We nevertheless proceed with the analysis assuming the validity of those coefficients in the retained lead term, and recognizing the risk. The ultimate test of the utility of the resultant analytic relations lies in the insights to which they lead and in the degree to which observations may be interpreted with their aid. Section 12 provides a preliminary test, and the test appears to be passed.

All R_j and T are now restricted to small values. The $c\{\delta m\}$ term in M, P and $G/2$ may be approximated as $1-(\delta m)^2/2 = 1-(\alpha_{mj}R_j - \beta_m T)^2/2$, while $s\{\delta m\}$ in Q becomes $(\alpha_{mj}R_j - \beta_m T) - (\alpha_{mj}R_j - \beta_m T)^3/6$. Higher-order terms could be retained, but they lead only to terms of higher order than the lead terms in the asymptotic expansion, and so are of minimal importance here.

The expansions required for M, P and Q depend on the wanted variable L and must be determined for each L in turn. They are inconvenient functions, but they simplify somewhat with the adoption of horizontally isotropic input spectra and of $T = 0$, simplifications that will be introduced shortly. For horizontal displacement s_h, vertical displacement s_3, horizontal velocity v_h and vertical velocity v_3, the retained lead coefficients are then as given in Appendix D by (D 1)–(D 4).

The expansion required for $G/2$, in contrast, is common to all L and so is given here. It initially contains four self-product terms, such as $R_1^2 \sum[(k_1S_{m1}+k_2S_{m2}+k_3S_{m3})^2\alpha_{m1}^2]/4$, and six cross-product terms such as $R_1R_2\sum[(k_1S_{m1}+k_2S_{m2}+k_3S_{m3})^2\alpha_{m1}\alpha_{m2}]/2$. The required sums over m will in general be different from one another, resulting in ten separate parameters that represent the Lagrangian spectrum. These can be reduced to a slightly more workable number by the assumption of horizontal isotropy, which we now impose. S_{m1} then becomes $S_{mh}\cos\psi_m$, S_{m2} becomes $S_{mh}\sin\psi_m$, and likewise with α_{m1} and α_{m2}, where ψ_m is the azimuth of propagation of the m mode (measured from the 1-axis). The results are then averaged over all ψ_m for a given α_{mh}, keeping S_{mh} constant. The spectrum is thereby characterized (for immediate purposes) by just seven sums, given by I_1 to I_7 in table 1. These sums, and others to follow, will hereinafter be termed 'spectral sums'.

Under present approximations, I_1 is the Lagrangian variance of vertical shear of horizontal velocity, normalized by N^2. It is therefore the inverse of a Richardson number Ri_L as obtained in Lagrangian coordinates. We may expect Ri_L to exceed 1, perhaps considerably. This is in fact a required condition for the linearity of the Lagrangian wave system (see Appendix A). In practice, $I_2 \ll I_1 \ll I_3$. Although defined as the Lagrangian variance of the horizontal velocity, I_4 under present approximations is equally N^2 times the variance I_{16} of the vertical displacement. All of I_1 to I_5 are inherently positive. Both I_6 and I_7 vanish if the input spectra are symmetric with respect to upgoing and downgoing waves (as in AJ89), but if all waves are upgoing only or downgoing only they will take the sign of the corresponding α_3.

In all of the spectral sums, we may imagine the summation operation $\sum$ being replaced by integration over α_h (> 0) and α_3 after deleting the m subscripts and multiplying by $2\pi\alpha_h\,\mathrm{d}\alpha_h\,\mathrm{d}\alpha_3$. In this process, the various expressions subject to summation become spectral densities in $\boldsymbol{\alpha}$ space.

Given these characterizing parameters, $G/2$ may be evaluated for horizontally

$$
\begin{aligned}
I_1 &\equiv \tfrac{1}{2}\sum S_h^2\alpha_h^2 = \tfrac{1}{2}\sum S_3^2\alpha_3^2 \\
&\qquad = \tfrac{1}{2}N^2\sum S_h^2\alpha_3^2\beta^2 \equiv [Ri_L]^{-1} \\
I_2 &\equiv \tfrac{1}{2}\sum S_3^2\alpha_h^2 \\
I_3 &\equiv \tfrac{1}{2}\sum S_h^2\alpha_3^2 = \tfrac{1}{2}\sum S_3^2\alpha_h^{-2}\alpha_3^4 \\
I_4 &\equiv \tfrac{1}{2}\sum S_h^2\beta^2 = \tfrac{1}{2}N^2\sum S_3^2 = N^2 I_{16} \\
I_5 &\equiv \tfrac{1}{2}\sum S_3^2\beta^2 = \tfrac{1}{2}N^2\sum S_3^2\alpha_h^2\alpha_3^{-2} \\
I_6 &\equiv \tfrac{1}{2}\sum S_h^2\alpha_3\beta = \tfrac{1}{2}N\sum S_3^2\alpha_h^{-1}\alpha_3^2\,\mathrm{sgn}[\alpha_3] \\
I_7 &\equiv \tfrac{1}{2}\sum S_3^2\alpha_3\beta = \tfrac{1}{2}N\sum S_3^2\alpha_h\,\mathrm{sgn}[\alpha_3] \\
I_8 &\equiv \tfrac{1}{2}\sum S_3^2\alpha_h^4 \\
I_9 &\equiv \tfrac{1}{2}\sum S_3^2\alpha_h^2\alpha_3^2 \\
I_{10} &\equiv \tfrac{1}{2}\sum S_3^2\alpha_3^4 \\
I_{11} &\equiv \tfrac{1}{2}\sum S_h^2\alpha_3^4 = \tfrac{1}{2}\sum S_3^2\alpha_h^{-2}\alpha_3^6 \\
I_{12} &\equiv \tfrac{1}{2}\sum S_3^2\alpha_h^2\alpha_3\beta = \tfrac{1}{2}N\sum S_3^2\alpha_h^3\,\mathrm{sgn}[\alpha_3] \\
I_{13} &\equiv \tfrac{1}{2}\sum S_3^2\alpha_3^3\beta = \tfrac{1}{2}N\sum S_3^2\alpha_h\alpha_3^2\,\mathrm{sgn}[\alpha_3] \\
I_{14} &\equiv \tfrac{1}{2}\sum S_h^2\alpha_3^3\beta = \tfrac{1}{2}N\sum S_3^2\alpha_h^{-1}\alpha_3^4\,\mathrm{sgn}[\alpha_3] \\
I_{15} &\equiv \tfrac{1}{2}\sum S_3^2\alpha_h^2\beta^2 = \tfrac{1}{2}N^2\sum S_3^2\alpha_h^4\alpha_3^{-2} \\
I_{16} &\equiv \tfrac{1}{2}\sum S_3^2 \\
I_{17} &\equiv \tfrac{1}{2}\sum S_h^2 = \tfrac{1}{2}\sum S_3^2\alpha_h^{-2}\alpha_3^2
\end{aligned}
$$

TABLE 1. Spectral sums required for the analysis. The summation is over all modes m as described in the text. The mode subscript m has been omitted from all mode-dependent quantities ($S_h, S_3, \alpha_h, \alpha_3$ and β) for simplicity. The alternative forms result from the standard approximations adopted in the text. Forms dependent only on S_3, α_h and α_3 are most useful for application when the spectrum is specified by $S_3 = S_3[\alpha_h, \alpha_3]$. Sgn$[\alpha_3]$ is the sign of α_3; spectral sums containing it vanish in the case of symmetric upgoing and downgoing spectra.

isotropic spectra as approximating $\hat{G}/2$, where

$$
\begin{aligned}
8\hat{G} \equiv\ & R_1^2(3k_1^2I_1 + k_2^2I_1 + 4k_3^2I_2) + R_2^2(k_1^2I_1 + 3k_2^2I_1 + 4k_3^2I_2) \\
&+R_3^2(4k_1^2I_3 + 4k_2^2I_3 + 8k_3^2I_1) + T^2(4k_1^2I_4 + 4k_2^2I_4 + 8k_3^2I_5) \\
&+4R_1R_2k_1k_2I_1 - 16R_1R_3k_1k_3I_1 - 16R_2R_3k_2k_3I_1 - 16R_1Tk_1k_3I_7 \\
&-16R_2Tk_2k_3I_7 + R_3T(8k_1^2I_6 + 8k_2^2I_6 + 16k_3^2I_7).
\end{aligned}
\tag{9.1}
$$

This set may be further simplified by setting $R_1 = w_1\cos\phi - w_2\sin\phi$, $R_2 = w_1\sin\phi + w_2\cos\phi$, $k_1 = k_h\cos\phi$ and $k_2 = k_h\sin\phi$, where ϕ is the azimuth of the $\boldsymbol{k}$ currently of interest (AJ89). It is then found that

$$
\hat{G}/2 = \varepsilon^2w_1^2 + \zeta^2w_2^2 + \eta^2R_3^2 + \theta^2T^2 - \kappa w_1R_3 - \lambda w_1T + \mu R_3T,
\tag{9.2}
$$

where

$$
\varepsilon^2 \equiv (3k_h^2I_1 + 4k_3^2I_2)/16, \quad \zeta^2 \equiv (k_h^2I_1 + 4k_3^2I_2)/16,
$$

$$
\eta^2 \equiv (k_h^2I_3 + 2k_3^2I_1)/4, \quad \theta^2 \equiv (k_h^2I_4 + 2k_3^2I_5)/4,
$$

$$
\kappa \equiv k_hk_3I_1, \quad \lambda \equiv k_hk_3I_7, \quad \mu \equiv k_h^2I_6/2 + k_3^2I_7.
$$

Simultaneously, the exponent in (5.2) reduces to $i(k_h w_1 + k_3 R_3 - \omega T)$. With no w_2 term in this exponent, and no cross-product term involving w_2 in (9.2), these expressions are simpler to employ. They have the further advantage of being in a form convenient for subsequent integration over k_h, rather than separately over k_1 and k_2.

Given this approximation for $G/2$, which is equivalent to that adopted by AJ89, and those already indicated for M, P and Q (rewritten in terms of w_1 and w_2 rather than R_1 and R_2), it can be seen that the integrations required in (5.2) are almost all of the form given by (7.4), with w there taken to be w_1, w_2, R_3 and T in turn. They are not all quite of that form because of cross-product terms in (9.2) involving $w_1 R_3$, $w_1 T$ and $R_3 T$, but (7.4) can be generalized by setting $w = v - c/2a^2$ to yield

$$\int dw\, w^n \exp[-a^2 w^2 + ibw - cw] = \exp[c^2/4a^2 - ibc/2a^2] \int dv\, (v - c/2a^2)^n \exp[-a^2 v^2 + ibv], \tag{9.3}$$

and then all the required integrals are available in closed form. (The equivalent step was taken in AJ89 by a further change of coordinate.)

The approximation (9.1) is inadequate to the purpose of obtaining an expansion that will yield correctly all terms of order k^{-3} in one-dimensional spectra, now that (in effect) the endpoint terms omitted from AJ89 are retained in $(M+P)$ and $(M-Q)$. Instead, terms of order $(\delta m)^4$ must be retained in (7.2) for that purpose. This poses something of a problem, since the closed-form integrals given by (7.4) and (9.3) rely on there being only terms up to second order in (δm) in the integrand exponent.

To circumvent this problem, we first expand (7.2) as

$$\begin{aligned} G/2 &= \sum (\boldsymbol{k} \cdot \boldsymbol{S}_m)^2 [(\delta m)^2/2 - (\delta m)^4/24 + \ldots]/2 \\ &= \hat{G}/2 - \sum (\boldsymbol{k} \cdot \boldsymbol{S}_m)^2 [(\delta m)^4/48 + \ldots]. \end{aligned} \tag{9.4}$$

We next insert this form into the exponentials wanted in (8.1) and (8.2), and finally expand the second exponential factor:

$$\begin{aligned} \exp[-G/2] &= \exp[-\hat{G}/2] \exp\left[\sum (\boldsymbol{k} \cdot \boldsymbol{S}_m)^2 (\delta m)^4/48 + \ldots\right] \\ &= \exp[-\hat{G}/2] \left[1 + \sum (\boldsymbol{k} \cdot \boldsymbol{S}_m)^2 (\delta m)^4/48 + \ldots\right]. \end{aligned} \tag{9.5}$$

This leaves the exponential proper in the form required for a closed-form integral, while introducing yet another factor 'on line' multiplying $(M + P)$ and $(M - Q)$. For purposes of the lowest-order retained asymptotic term, inspection reveals that the expansion may be truncated at the term shown explicitly in (9.5), to yield the correcting factor $(1 + F)$ with

$$F \equiv \sum (\boldsymbol{k} \cdot \boldsymbol{S}_m)^2 (\delta m)^4/48. \tag{9.6}$$

It turns out that this can be a significant contributor to the spectrum.

With the use of this factor, (8.1) and (8.2) are replaced by

$$A = A_P \doteq (M + P)(1 + F) \exp[-\hat{G}/2] \tag{9.7}$$

and

$$A = A_Q \doteq (M - Q)(1 + F) \exp[-\hat{G}/2], \tag{9.8}$$

respectively. These forms can be integrated as required in (5.2) by successive application of (9.3) with w taken to be w_1, w_2, R_3 and T in turn, in whatever sequence is found convenient. After removal of the term in k^{-3} as previously discussed, they yield accurate coefficients for the lowest-order retained term of the three-dimensional asymptotic expansion, the approximate nature of their derivation having implications only for higher-order terms.

In general, F contains a daunting array of individual terms even after averaging in azimuth. This array reduces to a workable number (in the present incompressible case) if attention is confined, as soon it will be, to the three-dimensional wavenumber spectrum. It then involves four further spectral sums – I_8 to I_{11}, found in table 1 – and is given in Appendix D by $\hat{F}$.

The product PF in (9.7) is redundant for purposes of the lowest-order retained term. It – but not QF in (9.8) – may be ignored.

Appendix D presents, in addition to $\hat{F}$, partially expanded, azimuthally averaged, forms for the $(M + P)(1 + \hat{F})$ of horizontal and vertical displacement, and for the $(M - Q)(1 + \hat{F})$ of horizontal and vertical velocity, to the extent required for the k^{-5} terms of three-dimensional spectra and so to the extent required for the k^{-3} terms of one-dimensional spectra. These various forms involve yet further spectral sums – I_{12} to I_{17} – found in table 1. They await integration over $\boldsymbol{R}$ space, to be accomplished via integration over w_1, w_2 and R_3.

10. Integral results: the four-, three- and one-dimensional asymptotic spectra

Closed-form integrals can be obtained in the manner described, but the process is tedious to an extreme. The tedium is reduced in the case of input spectra that are symmetric with respect to upgoing and downgoing waves. For them, five of the spectral sums vanish, and so too do the cross-product terms in w_1T and R_3T found in (9.2). This simplifies the T integrals and leads to Eulerian frequency spectra in the tail proportional to the Gaussian form $\exp[-\omega^2/4\theta^2]$. The factor of proportionality is itself a function of frequency and wavenumber derived from Hermite polynomials, and these must be determined individually for individual L variables.

It is possible to reduce (but by no means eliminate) the tedium further, even for vertically asymmetric spectra, if the frequency spectrum is ignored and interest is instead limited to the spatial three-dimensional and one-dimensional wavenumber spectra that are the prime focus here. This may be achieved by the simple expedient of invoking stationarity in time and setting $T = 0$; one $[2\pi]^{-1}$ factor is removed from (5.2), and the integration over T is ignored.

The triple integration over w_1, w_2 and R_3 then produces one factor common to all wave variables and a second factor specific to each of the various combinations of $w_1^x w_2^y R_3^z$ that are found in (D 1)–(D 4). The common factor, which includes also the $[2\pi]^{-3}$ factor left over in (5.2), is given as C in Appendix E. The others, given there as entries in table 2, are to be multiplied by C and inserted in place of $w_1^x w_2^y R_3^z$ in (D 1)–(D 4). These will then provide the k^{-5} terms of the respective three-dimensional spectra.

Observations normally produce one-dimensional spectra. These are to be found by integrating the three-dimensional spectra over the two unobserved wavenumber components. The integrations can be done accurately only by numerical methods. Approximate coefficients can be estimated, however, as outlined in the next section.

11. Approximate coefficients for one-dimensional vertical-wavenumber spectra

Vertical-wavenumber spectra are found by multiplying three-dimensional spectra by $2\pi k_h \, dk_h$ and integrating over k_h from 0 to infinity. The approximation adopted here is outlined in Appendix F. We apply it here only to the opening term of (D 2), and even then only for $I_1 \to 0$. This yields, as an approximation for the prototypical s_3 asymptotic spectrum,

$$[s_3]_{tail} = (32\pi)^{-1/2} I_1^{-3/2} \exp[-1/(2I_1)] |k_3|^{-3}. \tag{11.1}$$

The corresponding asymptotic spectrum of v_h is found from this by multiplying by N^2 (just as in linear waves, under current approximations). For s_h, the multiplier is I_3/I_1, and for v_3 it is $N^2 I_2/I_1$.

The coefficient of $|k_3|^{-3}$ attains a maximum under variations of I_1 at $I_1 = \frac{1}{3}$, where its value is 0.12. This maximum could be viewed as a 'saturation' of 'the -3 spectral tail' that occurs as the intensity of the input spectrum is increased; but unfortunately the approximation and the original Lagrangian linearization are of questionable legitimacy at such a large I_1. Nevertheless, some tendency to a saturation of this type as I_1 increases toward $\frac{1}{3}$ – i.e. as Ri_L decreases toward 3 – may be expected to occur. If so, its revelation must await numerical results (now pending).

12. Discussion

The mathematical demand that an Eulerian spectral tail should exist was first established by Allen & Joseph in 1989, most particularly with respect to the vertical displacement in a certain canonical spectrum. The form taken by the tail has been confirmed here, for general quasi-linear Lagrangian input spectra, though the present coefficients include contributions apparently omitted inadvertently from AJ89 (Appendix B), with the unacceptable lead term suppressed (Appendix C). The analysis has also been extended to treat a variety of wave variables, both those in phase with vertical displacement and those in phase quadrature (but with the omission of compressibility and Coriolis effects except as discussed in Appendix G).

We have found relations that can yield reasonably accurate four-, three- and one-dimensional spectra by numerical means, and crude analytic estimates of the one-dimensional vertical-wavenumber spectral tails for horizontal and vertical components of displacement and velocity. An important aspect of the one-dimensional behaviour, revealed by (11.1), is that the coefficient shared by $PSD[s_3]$ and $N^{-2}PSD[v_h]$ depends exclusively on a single spectral sum, I_1 – alternatively, on the Lagrangian Richardson number Ri_L – at the present level of approximation. Corrections to it, whether obtained analytically or numerically, depend on certain 'shape factors' (such as $I_2 I_3/I_1^2$), but these tend to be close to 1 in model spectra that have been tested (or = 0, in some cases, with vertical symmetry) and so are unlikely to alter severely the Eulerian spectral tails from one model input spectrum to another. Universality of saturation conditions would then depend only on the universality of some defining Ri_L.

We have seen that (11.1) exhibits saturation at a value $Ri_L = 3$, the saturated value being 0.12 $|k_3|^{-3}$ for s_3, and so 0.12 $N^2|k_3|^{-3}$ for v_h. The coefficient 0.12 may be compared: with the most commonly cited atmospheric value, namely 0.3 (taken from v_h spectra, e.g. Tsuda *et al.* 1989, once the two horizontal components of velocity are combined); with the value 0.4 obtained from representative oceanic s_3 spectra (derived from Gregg 1977, with change of cycle units to radian units); and with the value

0.15 inferred from oceanic velocity-shear spectra (sloping straight line in figure 3 of Gargett *et al.* 1981, with change to radian units). The potential for an explanation of universality of saturated spectra, at observed intensities, may well be in sight.

The potential for saturation in this fashion, at observed intensities, may not be confirmed by accurate numerical evaluations (nor perhaps for Ri_L somewhat larger, as would be wanted to validate Lagrangian linearization). However, another limiting process, ignored to this point, must come into play in practice: dissipation, notably as induced by instability of the wave system. It too is under the control of Ri_L, and in turn can control Ri_L, in a self-limiting fashion.

Observed and model spectra, both in the atmosphere and in the oceans, exhibit an Eulerian Richardson number Ri_E of 1 or less, implying instability and associated dissipation. A substantial portion of the shear that produces this result derives from the -3 portion of the respective spectra. (Gargett *et al.* 1981, quoting an argument of Walter Munk for oceanic systems, conclude that the Eulerian Richardson function decreases to 1 just as $|k_3|$ rises to the point of onset of the -3 portion of the spectrum. Further reduction, to some effectively destabilizing value of Ri_E, then depends entirely on the -3 portion of the spectrum. Hines 1991*a* finds a similar bias towards the -3 portion for the production of instability in a standard atmospheric spectral model.) This dependence implies that the -3 portion of the tail must rise to a sufficient intensity as a prerequisite for significant dissipation. That in turn suggests that Ri_L must be reduced to values of 10 or less to become self-limiting, as follows.

When $Ri_L > 15$ or so, the Eulerian and Lagrangian spectra would be much the same over the spectral region observed, and Ri_E as determined for that region would approximate to Ri_L. However, with reduction of Ri_L from 15, the Eulerian spectrum contributes significant – and increasingly significant – extra shears from the tail region: Ri_E must reduce more rapidly than Ri_L. As Ri_E decreases, an increasing fraction of space–time becomes unstable and dissipation becomes more rapid. Some value in the range $4 < Ri_L < 10$, perhaps $Ri_L = 7$, may be suggested as the value required to dissipate wave energy at a rate equal to the input rate. We are therefore led to the concept of a self-limiting, quasi-universal Ri_L near 7, accompanied by a quasi-universal Ri_E that is but a fraction (perhaps a small fraction) of 1.

We postpone further discussion until numerical results, now pending, become available for presentation.

This study has evolved over a long period, with input from many individuals along the way. Throughout, discussions with Dr Igor Chunchuzov have proved to be most valuable. He was conducting a parallel analysis by methods of his own, and repeated exchanges between us were of great benefit, certainly to me. Others who have provided useful discussion and often helpful input, while not necessarily agreeing with the course adopted, are: R. Akmaev, R. N. Bracewell, J. B. Kinney, G. Klaassen, H. G. Mayr, A. Medvedev, K. Omidvar, R. E. Peltier, L. R. O. Storey and M. P. Sulzer. L. Childress and J. B. Kinney, working with Dr Sulzer at the Arecibo Observatory, programmed and conducted the numerical computations referenced. The work was supported in part by the Laboratory for Atmospheres of NASA's Goddard Space Flight Center via Furman University and by the National Science Foundation via the National Astronomy and Ionosphere Center at Cornell University.

Appendix A. Conditions for quasilinearity of the Lagrangian spectrum

Throughout this paper, it is assumed that the Lagrangian waves can be represented by dispersion and polarization relations taken from linear theory. The assumption follows from a 'weakly nonlinear' approximation to the full Lagrangian equations, although with respect to temporal spectra it must also be assumed that the decorrelation time scale in such an approximation exceeds the times T required in § 9 (AJ89).

The conditions for validity are derived from the underlying dynamical equations, and are developed in some detail by Hines (2002). They are based on a requirement that the linear dispersion and polarization relations that result from the omission of nonlinear terms would not be seriously altered if representative r.m.s. values for the nonlinear terms were to be introduced.

The requisite conditions parallel a number of conditions that apply to Eulerian spectra for linearity. The best known of these is that $\langle v_j\rangle|k_j|$ shall be small in comparison with wave frequency, so that Doppler spreading in a wave of wavevector $\boldsymbol{k}$ should be of no great importance – the very condition that breaks down in practice and leads to the present analysis.

An essential prerequisite for Lagrangian quasi-linearity is found to be that the Lagrangian Richardson number Ri_L, defined as N^2 divided by the variance of vertical shear of horizontal velocity and given by I_1^{-1} here, shall be $\gg 1$. This is a condition imposed on any broad spectrum as a whole, quite independently of its application to one wave or another. (A corresponding condition, $Ri_E \gg 1$, must be met for Eulerian linearity.)

A number of other conditions obtain, but in practice they can all be subsumed into the foregoing or into one of two others, both of which are wave-specific. The first requires that the r.m.s. horizontal gradient of vertical displacement should be $\ll |\alpha_h/\alpha_3|$ for the wave in question, which quantity is given by β/N under the approximations employed here. (A corresponding condition must be met for Eulerian linearity.) The second requires that the r.m.s. vertical shear of horizontal displacement should be $\ll |\alpha_3/\alpha_h|$, or N/β, for the wave in question. (There is no corresponding Eulerian condition. It is replaced, in the Eulerian formulation, by the Doppler-spread condition already cited.)

Model spectra adopted for use with the present analysis should consist only of modes that meet or come close to meeting these two conditions, and the spectrum as a whole should satisfy or come close to satisfying $Ri_L \gg 1$.

For further discussion of the conditions themselves, see Hines (2002).

Appendix B. The terms missing from AJ89

The forms given by (8.1) and (8.2) lead inevitably to a k^{-1} term as the lead term in an asymptotic expansion (at large k) in the one-dimensional Eulerian spectral tail, when the integrations over $\boldsymbol{R}$ and T are conducted following the procedures of AJ89. In contrast, AJ89 came to the conclusion that the lead term for one-dimensional spectra would be of the form k^{-3}. The purpose of this appendix is to identify the point at which AJ89 missed out the makings of the k^{-1} term, and to establish the invalidity of its having done so. The coefficients of higher-order terms are also at stake.

The point of loss came in the transition from (3.6) to (3.7) of AJ89. This transition was stated to be an integration by parts, but it ignored the endpoint term that must in principle arise in such an integration. Inclusion of this term would have produced the k^{-1} term of the one-dimensional spectrum and also altered the k^{-3} term that was in fact found. It should have been included, if valid coefficients were to be had, even though its k^{-1} term must be eliminated for unrelated reasons (see Appendix C).

To simplify the proof, we introduce an ordering parameter p, to which all Lagrangian wave amplitudes are proportional. We concentrate on contributions made by the lowest-order terms in p as $p \to 0$, since they exhibit the error and cannot be offset by terms of higher order. Time variations play no part in the discussion, so they are omitted from the notation. Moreover, the difference between the incompressibility condition $J = 1$ (assumed in the main text here) and the more general $J \neq 1$ (permitted by AJ89) concerns only higher-order terms in p and so becomes irrelevant to the discussion.

The Eulerian waveform $E[\boldsymbol{x}]$ was to be obtained from the Lagrangian waveform $L[\boldsymbol{r}]$ by multiple integrations found in (3.6) of AJ89. The first integration of relevance here is integration over Lagrangian coordinate r_α of $\varepsilon_{\alpha\beta\gamma}(L[\boldsymbol{r}])(\partial E_1/\partial r_\alpha)(\partial E_2/\partial r_\beta)(\partial E_3/\partial r_\gamma)$, with triple summation over α, β and γ (= 1, 2 and 3), where $\varepsilon_{\alpha\beta\gamma}$ is the perfectly antisymmetric unit tensor of the third rank and

$$E_q \equiv \exp[-\mathrm{i}m_q\{r_q + s_q(\boldsymbol{r})\}] \tag{B 1}$$

(with no summation over q). The lead term of this integrand, under the p ordering, arises from the contribution to the triple sum from the term in which $\alpha = 1$, $\beta = 2$ and $\gamma = 3$. It is given by $-\mathrm{i}^3 m_1 m_2 m_3 L[\boldsymbol{r}] \exp[-\mathrm{i}\boldsymbol{m}\boldsymbol{\cdot}\boldsymbol{r}]$. Aside from the extraneous factor $m_2 m_3 \exp[-\mathrm{i}m_2 r_2 - \mathrm{i}m_3 r_3]$, then, the wanted integral (over some range $R_L \leqslant r_1 \leqslant R_U$) is

$$\begin{aligned} \mathrm{i}m_1 \int_{R_L}^{R_U} \mathrm{d}r_1\, L[r_1, r_2, r_3] \exp[-\mathrm{i}m_1 r_1] &= -\int_{r_1=R_L}^{r_1=R_U} \mathrm{d}(\exp[-\mathrm{i}m_1 r_1]) L[r_1, r_2, r_3] \\ &= \int_{R_L}^{R_U} \mathrm{d}r_1 (\partial L[\boldsymbol{r}]/\partial r_1) \exp[-\mathrm{i}m_1 r_1]) + \{-L[r_1, r_2, r_3] \exp[-\mathrm{i}m_1 r_1]\}_{R_L}^{R_u}, \end{aligned} \tag{B 2}$$

in which $\{\ldots\}_{R_L}^{R_U}$ denotes the value at R_U minus the value at R_L, which difference we denote as W. It is this endpoint term (and corresponding terms of higher order in p) that AJ89 omitted, for no stated reason.

The boundary conditions imposed by AJ89 were periodic, and it may be supposed that W was taken to vanish via cancellation of the two terms in it. However, such a treatment would ignore the fact that a second integration was required, one that introduced an infinite coefficient. Greater delicacy is required.

The relevant integration, found in (3.6) of AJ89, demands multiplication of W by $\mathrm{d}m_1\, m_1^{-1} \exp[\mathrm{i}m_1 x_1]$ and then integration over m_1 through the range from $-\infty$ to $+\infty$, including of course $m_1 = 0$, where the infinite coefficient arises. This produces, as the contribution from R_U,

$$-L[R_U, r_2, r_3] \int \mathrm{d}m_1\, m_1^{-1} \exp[\mathrm{i}m_1(x_1 - R_U)] = -\mathrm{i}\pi L[R_U, r_2, r_3]\, \mathrm{sgn}[x_1 - R_U], \tag{B 3}$$

in which sgn is the signum function: sgn is -1 if its argument is negative and $+1$ if its argument is positive (and 0 if its argument is 0). There is a corresponding contribution from the lower endpoint. For $x_1 < R_L$, both signum functions are negative, and subtraction of the R_L contribution from the R_U contribution produces the anticipated cancellation once periodic boundary conditions are imposed (such that $L[R_U, r_2, r_3] = L[R_L, r_2, r_3]$). Similarly for $x_1 > R_U$. In the intermediate range

$R_L < x_1 < R_U$, however, the signum functions are of opposite signs and the two endpoint terms add, rather than cancel. It is this addition of endpoint terms, in place of the cancellation that usually arises with periodic boundary conditions, that was overlooked in AJ89.

That the additional term must be included (if one insists on proceeding via integration by parts) can be seen readily by choosing $L = 1$ throughout the domain of integration $R_L < r_1 < R_U$. The terms retained by AJ89 are all proportional to spatial derivatives of L, and so would have produced, as the corresponding Eulerian function, $E = 0$. Inclusion of the endpoint term leads to the correct Eulerian function, $E = 1$, over the domain of interest. (This function could also be produced by the inclusion of delta-function derivatives at the endpoints, in lieu of the endpoint term.) Both the present paper and that of Chunchuzov (2002) avoid the issue entirely, simply by avoiding the integration by parts. In doing so, both yield the k^{-1} one-dimensional asymptotic term and alter the coefficients of higher-order terms from the values that would be obtained by AJ89, absent the endpoint term.

The newly introduced k^{-1} terms are unacceptable both mathematically and physically, because of the infinite variances they produce. However, they are properly removed, not simply by ignoring the endpoint term, but rather by the adoption of a corrective procedure (see Appendix C). The end result is the same for the k^{-1} terms – namely, their elimination – but the coefficients of the k^{-3} terms are dependent on the procedure actually employed, absent a rigorously correct specification. The procedure adopted in Appendix C is suggested as being preferable to that implicit in AJ89, which was produced simply by oversight.

Appendix C. Elimination of infinite Eulerian variances

As noted in § 9, a straightforward application of the AJ89 protocol would produce in the present analysis infinite Eulerian variances. These are inadmissible as valid consequences of the analysis, and an explanation for their occurrence should be given to validate their removal.

For in-phase variables, the inadmissible terms derive solely from the lead term of M, namely $\sum L_m^2/2$, a constant. This fact leads us to examine the Lagrange → Euler transformation of a constant: i.e. we now adopt $L = L_0$, a constant (e.g. unperturbed temperature). The steps taken in §§ 2–10 for fluctuating quantities may be repeated. They lead in due course to

$$A = A_0 = M_0 \exp[-G/2] \tag{C 1}$$

in place of (8.1) and (8.2), with $M_0 = L_0^2$. The Eulerian spectrum of L_0 is then to be obtained, as in (5.2), as a multiple of the three-dimensional spatial integral $\int d^3R \exp[-G/2 + i\boldsymbol{k} \cdot \boldsymbol{R}]$; but we know in advance that this spectrum, being the spectrum of a constant, must vanish at all $\boldsymbol{k}$ except at $k = 0$, where it has a singularity. It follows that:

$$\int d^3R \exp[-G/2 + i\boldsymbol{k} \cdot \boldsymbol{R}] = 0. \quad (k \neq 0). \tag{C 2}$$

This is an integral that was supposed to be evaluated in the main text after approximating G by $\hat{G}$, in accordance with the AJ89 protocol. With that approximation it would in fact have produced a k^{-3} three-dimensional spectrum, a k^{-1} one-dimensional spectrum. The conclusion is inevitable: the approximation is inadequate to the need in the case of a constant coefficient, whether it be L_0^2 as here, or $\sum L_m^2/2$ as in the main text.

The nature of the failure of the approximation seems equally clear. Although $\exp[-G/2]$ decreases rapidly from 1 with increase of any R_j when k is chosen to be large, as argued in AJ89, it nevertheless rises again, to values comparable to 1, here and there throughout the infinite domain over which the integration of $\exp[-G/2+\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{R}]$ is to be conducted. It does so at places where a great many individual $c\{\delta m\}$ factors by accident happen to be nearly in phase, just as all are at $\boldsymbol{R}=0$. These additional contributions – and, indeed, the lesser contributions from the remaining portions of the infinite domain – must act to cancel the contribution obtained near $\boldsymbol{R}=0$ in order to produce the correct result, 0. (Indeed, G itself may require some correction to produce this result rigorously; its form was established only with the adoption of approximations.) The AJ89 argument that G may be employed to produce a cutoff when k is large, and so may itself be approximated by a quadratic form as in (9.1), is seen to be fallible.

In one sense, the failure is not great; the approximation leads to a spectral contribution at large k that is very small – small, in that k^{-1} is small at large k – but it does not quite provide the correct answer; the contribution is so small that it is in fact precisely 0. We apply this conclusion to the wave-related M of (6.1), and so delete the spectral terms that appear to imply infinite variances of in-phase variables.

Phase-quadrature variables suffer from the same problem and must be corrected in the same way. When the input Lagrangian spectrum is not symmetric with respect to upgoing and downgoing waves, these variables lead to a further contribution (from Q) that produces infinite variances. No formal proof of their illegitimacy analogous to that for M has yet been found, but the illegitimacy itself may be inferred as before from the infinite Eulerian variances themselves. With the fallability of the AJ89 protocol already established at order k^{-1} in one-dimensional spectra, we simply ignore the terms derived from Q that imply infinite variances. (The search for a formal proof continues.)

It might be thought that the entire problem arose here because of the present, somewhat aphysical, assumption of incompressibility. This cannot be the case, as renewed application of the p ordering of Appendix B would reveal. Nevertheless, the nonlinear terms contained in J do impose an upper limit on the k values to which the present analysis can claim validity – a limit that effectively excludes the k^{-1} term, see Appendix G.

Appendix D. Azimuth-averaged functions of w_1, w_2 and R_3

Attention is confined here to relations required for the three-dimensional wave-number spectra alone, obtained by setting $T=0$ everywhere and deleting from (5.2) one $[2\pi]^{-1}$ factor and the integration over T. These relations provide factors for horizontal and vertical components of displacement (s_h, s_v) and of velocity (v_h, v_v), which are representative of in-phase and phase-quadrature fluctuations, respectively. They require multiplication by $\exp[-\hat{G}/2]$ and insertion in place of A in the spatial three-dimensional equivalent of (5.2), for subsequent integration over all $\boldsymbol{R}$ to yield the lowest-order (i.e. k^{-5}) retained component of the respective Eulerian three-dimensional tail spectra.

The s_h component is obtained by adding the components found for s_1 and s_2 (these being contributions to variance). It is

$$[s_h]_{-5} = -\tfrac{1}{4}(I_1 w_*^2 + 2I_3 R_3^2) + \hat{F} I_{17} + k_3^2 I_1^2 w_*^2 R_3^2/4$$
$$-k_3 k_h I_1(3w_1 w_*^2 R_3/16 + I_3 w_1 R_3^3/4)$$
$$+k_h^2(I_1^2 w_1^2 w_*^2/2 + I_1^2 w_*^4/16 + I_1 I_3[3w_1^2 + w_2^2]R_3^2/2 + I_3^2 R_3^4), \qquad \text{(D 1)}$$

in which $w_*^2 \equiv w_1^2 + w_2^2$ and $\hat{F}$ is as given in (D 5). In use, all terms must be expanded (if necessary) into the form $K_{xyz} w_1^x w_2^y R_3^z$. After the required multiplication and insertion in the three-dimensional equivalent of (5.2), integration over $\boldsymbol{R}$ yields results given in Appendix E. Likewise in what now follows.

The vertical displacement, whose spectrum was sought by AJ89, yields

$$[s_3]_{-5} = -\tfrac{1}{4}(I_2 w_*^2 + 2I_1 R_3^2) + \hat{F} I_{16} + \tfrac{1}{16}(k_3[I_2 w_*^2 + 2I_1 R_3^2] - 2k_h I_1 w_1 R_3)^2. \qquad \text{(D 2)}$$

The final factor will require its squaring operation to be conducted explicitly and, as before, expansion into terms of the form $K_{xyz} w_1^x w_2^y R_3^z$.

The horizontal velocity v_h (obtained as was s_h) produces

$$[v_h]_{-5} = -(N^2/4)(I_2 w_*^2 + 2I_1 R_3^2) + k_3^2 I_7(I_{12} w_*^4/32 + I_{13} w_*^2 R_3^2/8)$$
$$-k_h k_3([I_6 I_{12} + 3I_7 I_{13})w_1 w_*^2 R_3/32 + [3I_6 I_{13} + I_7 I_{14}]w_1 R_3^3/24)$$
$$+k_h^2 I_6(I_{13}[3w_1^2 + w_2^2]R_3^2/32 + I_{14} R_3^4/24)$$
$$+\hat{F}[I_4 - \tfrac{1}{4}(k_3^2 I_7^2 w_*^2 - 2k_3 k_h I_6 I_7 w_1 R_3 + k_h^2 I_6^2 R_3^2)], \qquad \text{(D 3)}$$

and the vertical velocity v_3 produces

$$[v_3]_{-5} = -(I_{15} w_*^2/4 + N^2 I_2 R_3^2/2) + k_3^2 I_7(I_{12} w_*^2 R_3^2/2 + I_{13} R_3^4/3)$$
$$-k_3 k_h I_7(3I_{12} w_1 w_*^2 R_3/8 + 2I_{13} w_1 R_3^3/3)$$
$$+k_h^2 I_7(I_{12} w_1^2 w_*^2/16 + I_{13} w_1^2 R_3^2/4)$$
$$+\hat{F}[I_5 - I_7^2(k_3^2 R_3^2 - k_h k_3 w_1 R_3 + k_h^2 w_1^2/4)]. \qquad \text{(D 4)}$$

It may be noted that, in vertically symmetric spectra (for which $I_6 = 0 = I_7$), $[v_h]_{-5}$ and $[v_3]_{-5}$ are greatly reduced in complexity.

In these,

$$48\hat{F} \equiv k_3^2(3I_8 w_*^4/4 + 6I_9 w_*^2 R_3^2 + 2I_{10} R_3^4) - k_3 k_h(6I_9 w_1 w_*^2 R_3 + 8I_{10} w_1 R_3^3)$$
$$+k_h^2(I_9 w_1^2 w_*^2/2 + I_9 w_*^4/8 + 9I_{10} w_1^2 R_3^2/2 + 3I_{10} w_2^2 R_3^2/2 + I_{11} R_3^4). \qquad \text{(D 5)}$$

(Note the factor 48.) Multiplications must be completed in the case of (D 3) and (D 4), and all terms require expansion to the form $K_{xyz} w_1^x w_2^y R_3^z$.

The foregoing relations are complete, and are intended for purposes of numerical evaluation of the asymptotic spectra. When one is interested only in the one-dimensional vertical-wavenumber asymptotic spectra, useful approximations can be obtained by setting $k_h = 0$ in these relations (but not in an associated exponent, $-B$), for reasons explained in Appendix F. This provides considerable simplification and permits some further analytic advance.

Appendix E. Results of integrations over w_1, w_2 and R_3

Appendix D provides in (D 1)–(D 4) terms that, after multiplication by the common factor $[2\pi]^{-3}\exp[-\hat{G}/2 - \mathrm{i}k_h w_1 - \mathrm{i}k_3 R_3]$ and integration over $\boldsymbol{R}$ space, will produce the

x, y, z	Associated multiplier of C
2,0,0	$\frac{1}{2}\hat{\varepsilon}^{-2}(1-\hat{H}^2/2)$
0,2,0	$\frac{1}{2}\zeta^{-2}$
1,0,1	$\frac{1}{4}\hat{\varepsilon}^{-2}\eta^{-2}(-\hat{k}_h k_3+\kappa[1-\hat{H}^2/2])$
4,0,0	$\frac{3}{4}\hat{\varepsilon}^{-4}(1-\hat{H}^2+\hat{H}^4/12)$
0,4,0	$\frac{3}{4}\zeta^{-4}$
2,2,0	$\frac{1}{4}\hat{\varepsilon}^{-2}\eta^{-2}(1-\hat{H}^2/2)$
2,0,2	$\frac{1}{4}\hat{\varepsilon}^{-2}\eta^{-2}(1-\hat{H}^2/2)(1-k_3^2/2\eta^2)-\frac{3}{8}\hat{\varepsilon}^{-4}\eta^{-4}\hat{k}_h k_3\kappa(1-\hat{H}^2/6)$
	$+\frac{3}{16}\hat{\varepsilon}^{-4}\eta^{-4}\kappa^2(1-\hat{H}^2+\hat{H}^4/12)$
3,0,1	$\frac{3}{8}\hat{\varepsilon}^{-4}\eta^{-2}\kappa(1-\hat{H}^2+\hat{H}^4/12)-\frac{3}{8}\hat{\varepsilon}^{-4}\eta^{-2}\hat{k}_h k_3(1-\hat{H}^2/6)$
1,0,3	$\frac{3}{8}\varepsilon^{-2}\hat{\eta}^{-4}\kappa(1-\hat{J}^2+\hat{J}^4/12)-\frac{3}{8}\varepsilon^{-2}\hat{\eta}^{-4}k_h\hat{k}_3(1-\hat{J}^2/6)$
1,2,1	$\frac{1}{8}\hat{\varepsilon}^{-2}\zeta^{-2}\eta^{-2}(-\hat{k}_h k_3+\kappa[1-\hat{H}^2/2])$
6,0,0	$\frac{15}{8}\hat{\varepsilon}^{-6}(1-3\hat{H}^2/2+\hat{H}^4/4-\hat{H}^6/120)$
0,6,0	$\frac{15}{8}\zeta^{-6}$
4,2,0	$\frac{3}{8}\hat{\varepsilon}^{-4}\zeta^{-2}(1-\hat{H}^2+\hat{H}^4/12)$
4,0,2	$\frac{3}{8}\hat{\varepsilon}^{-4}\eta^{-2}(1-\hat{H}^2+\hat{H}^4/12)(1-k_3^2/2\eta^2)-\frac{15}{16}\hat{\varepsilon}^{-6}\eta^{-4}\hat{k}_h k_3\kappa(1-\hat{H}^2/3+\hat{H}^4/60)$
	$+\frac{15}{32}\hat{\varepsilon}^{-6}\eta^{-4}\kappa^2(1-3\hat{H}^2/2+\hat{H}^4/4-\hat{H}^6/120)$
2,4,0	$\frac{3}{8}\hat{\varepsilon}^{-2}\zeta^{-4}(1-\hat{H}^2/2)$
2,2,2	$\frac{1}{8}\hat{\varepsilon}^{-2}\zeta^{-2}\eta^{-2}(1-\hat{H}^2/2)(1-k_3^2/2\eta^2)+\frac{3}{16}\hat{\varepsilon}^{-4}\zeta^{-2}\eta^{-4}\hat{k}_h k_3\kappa(1-\hat{H}^2/6)$
	$+\frac{3}{32}\hat{\varepsilon}^{-4}\zeta^{-2}\eta^{-4}\kappa^2(1-\hat{H}^2+\hat{H}^4/12)$
3,2,1	$\frac{3}{16}\hat{\varepsilon}^{-4}\zeta^{-2}\eta^{-2}\kappa(1-\hat{H}^2+\hat{H}^4/12)-\frac{3}{16}\hat{\varepsilon}^{-4}\zeta^{-2}\eta^{-2}\hat{k}_h k_3(1-\hat{H}^2/6)$
3,0,3	$\frac{9}{16}\hat{\varepsilon}^{-4}\eta^{-4}\kappa(1-\hat{H}^2+\hat{H}^4/12)-\frac{9}{16}\hat{\varepsilon}^{-4}\eta^{-4}\hat{k}_h k_3(1-\hat{H}^2/6)(1-k_3^2/6\eta^2)$
	$+\frac{45}{64}\hat{\varepsilon}^{-6}\eta^{-6}\hat{k}_h k_3\kappa^2(1-\hat{H}^2/3+\hat{H}^4/60)$
	$+\frac{15}{64}\hat{\varepsilon}^{-6}\eta^{-6}\kappa^3(1-3\hat{H}^2/2+\hat{H}^4/4-\hat{H}^6/120)$
5,0,1	$\frac{15}{16}\hat{\varepsilon}^{-6}\eta^{-2}\kappa(1-3\hat{H}^2/2+\hat{H}^4/40-\hat{H}^6/120)-\frac{15}{16}\hat{\varepsilon}^{-6}\eta^{-2}\hat{k}_h k_3(1-\hat{H}^2/3+\hat{H}^4/60)$

TABLE 2. Multipliers of C obtained upon integration over w_1, w_2 and R_3 of $w_1^x w_2^y R_3^z$ for various x, y and z combinations found in the first column. Missing members (such as 0,0,4) are to be obtained by interchanging x and z and replacing $\hat{\varepsilon}$ by $\hat{\eta}$, ε by η, $\hat{k}_h$ by $\hat{k}_3$, k_3 by k_h and $\hat{H}$ by $\hat{J}$ in the manner illustrated by 3,0,1 and 1,0,3.

lead term of the wanted asymptotic expansion for the three-dimensional wavenumber spectra (in radian units). These terms are all of the form $K_{xyz}w_1^x w_2^y R_3^z$, in which K_{xyz} is a function of k_h, k_3 and the spectral sums, while x, y and z take on integral values ranging from 0 to 6. Integration over w_2 is relatively straightforward, because of the absence of w_2 from any cross-product term or imaginary term in the exponent. Integration over w_1 and R_3 is preferably done first with respect to the variable of lower order – for example, with respect to w_1 for the $w_1R_3^3$ term – since this requires fewer terms in the expansion of $(v-c/2a^2)^n$ in (9.3).

All integrations yield a common factor, given by

$$C \equiv 2^{-3}\pi^{-3/2}|\varepsilon^{-1}\zeta^{-1}\hat{\eta}^{-1}|\exp[-B] \equiv 2^{-3}\pi^{-3/2}|\hat{\varepsilon}^{-1}\zeta^{-1}\eta^{-1}|\exp[-B], \qquad \text{(E 1)}$$

where

$$\hat{\varepsilon}^2 \equiv \varepsilon^2 - k^2/4\eta^2, \quad \hat{\eta}^2 \equiv \eta^2 - k^2/4\varepsilon^2, \tag{E 2}$$

and

$$B \equiv (\varepsilon^2 k_3^2 + \kappa k_h k_3 + \eta^2 k_h^2)/(4\varepsilon^2\eta^2 - k^2). \tag{E 3}$$

The factor $[2\pi]^{-3}$ associated with radian units has been incorporated in C.

Coefficients for terms up to a combined total of sixth order in w_1, w_2 and R_3 are required. They are listed in table 2 for the required x, y and z combinations. There,

$$\hat{k}_h \equiv k_h + \kappa k_3/2\eta^2, \quad \hat{k}_3 \equiv k_3 + \kappa k_h/2\varepsilon^2, \tag{E 4}$$

$$\hat{H}^2 \equiv \hat{k}_h^2/\hat{\varepsilon}^2, \quad \hat{J}^2 \equiv \hat{k}_3^2/\hat{\eta}^2. \tag{E 5}$$

These various coefficients may be inserted directly into (D 1)–(D 4) in place of the various $w_1^x w_2^y R_3^z$ found there after all expansions are complete. After this insertion, and then multiplication by C, (D 1)–(D 4) provide the k^{-5} terms of the respective asymptotic expansions of the three-dimensional spectra.

Even now, a daunting task lies ahead before values for the coefficients of these terms can be determined. In reality, they can be obtained analytically only for special spectra, and for general spectra only by numerical means. The forms given here are fully correct for either purpose, requiring only that the choice of input spectra be made.

Appendix F. Approximate method of integration over k_h

The k_3 spectra are produced from the three-dimensional spectra by integration over k_h (from 0 to ∞), after multiplication by $2\pi k_h\,\mathrm{d}k_h$. The approximation to be employed hinges on the fact that the factor $\exp[-B]$ in (E 1) falls rapidly toward 0 as k_h increases from 0, providing a cutoff behaviour somewhat similar to that which permitted evaluation of the asymptotic terms themselves. This is seen from the Taylor expansion of B (for k_h^2/k_3^2 small), whose first two terms are

$$\hat{B} \equiv (2I_1)^{-1} + (4k_h^2/I_2k_3^2)(1 - \{I_2I_3/4I_1^2\}/16). \tag{F 1}$$

The cluster $\{I_2I_3/I_1^2\}$ is typically about 1.4, at least in representative atmospheric spectra, and so the factor multiplying $(4k_h^2/I_2k_3^2)$ is typically only slightly less than 1. For simplicity, we take it to be 1. On the other hand, I_2 is typically of order $I_1/100$, while I_1 is necessarily $\ll 1$ for the presumed Lagrangian linearity to be available. In consequence, I_2 is extremely small, the second term in (F 1) increases rapidly with $|k_h/k_3|$ as the integration over k_h proceeds, and $\exp[-B]$ decreases rapidly toward 0.

These considerations lead to an approximation such that table 2 is evaluated only at $k_h = 0$. This choice eliminates many terms and simultaneously simplifies the evaluation of ε, ζ and η. We may now imagine a new table, produced from table 2 by evaluation at $k_h = 0$.

The components (D 1)–(D 4) appropriate to the various wave parameters are to be multiplied by $\pi\exp[-\hat{B}]\,\mathrm{d}(k_h^2)$ and integrated over k_h^2 from 0 to infinity. The integration produces a common factor $D|k_3|^{-1}$, with

$$D \equiv (8\pi)^{-1/2} I_1^{-1/2} \exp[-1/(2I_1)]. \tag{F 2}$$

This common factor is to be employed much as C was in Appendix E. Entries from the imagined table are to be inserted in (D 1)–(D 4) and then multiplied by

$D|k_3|^{-1}$ to produce the lead term of the one-dimensional asymptotic expansions of the respective wave variables. (A further factor of k_3^{-2} is contributed from within the imagined table.) A factor of 2 has been incorporated into D to effect a combination of positive and negative k_3 into a single-sided $|k_3|$ spectrum for comparison with standard observations.

The cutoff, such as it is, occurs when k_h^2 increases to values exceeding $I_2 k_3^2/4$. Examination of the entries in table 2 reveals that one term dominates over all others (the more so, as $I_1 \to 0$) when k_h^2 is much less than this value. There are correction terms even at $k_h = 0$, but they are not significant unless the omitted terms in k_h are also significant before cutoff occurs. Accordingly, the approximation with $k_h = 0$ is valid only so long as the contribution made by the dominant term itself remains dominant.

The dominant term is provided by the R_3^2 part of the first term in each of (D 1)–(D 4), and more particularly by the contribution from the $\hat{J}^2$ term in the implied entry in table 2. It leads to the results given in the opening paragraph of § 11.

Appendix G. Compressibility and Coriolis effects

Both compressibility and Coriolis effects would introduce complications into the analysis, specifically through modified dispersion and polarization relations. Compressibility also imposes a limitation on applicability.

Compressibility requires the retention of the Jacobian $J[\boldsymbol{s}]$ that appeared in (3.2) but was subsequently dropped. Its form is

$$J[\boldsymbol{s}] = \sum\sum\sum \varepsilon_{\alpha\beta\gamma}(\delta_{1\alpha} + s_{1,\alpha})(\delta_{2\beta} + s_{2,\beta})(\delta_{3\gamma} + s_{3,\gamma}), \tag{G 1}$$

in which $\varepsilon_{\alpha\beta\gamma}$ is the perfectly antisymmetric unit tensor of the third rank, δ_{ij} is the Kronecker delta, a comma indicates differentiation with respect to the coordinate index that follows it, and the summations are over α, β and $\gamma = 1, 2$ and 3 (Lamb 1945, or see AJ89). This form combines with the factor $\exp[-i\boldsymbol{K}\cdot(\boldsymbol{r}+\boldsymbol{s})]$ of (3.4) to produce the triple sum described in Appendix B prior to (B 1), written there with $\boldsymbol{m}$ replacing $\boldsymbol{K}$. Although the result provided AJ89 with a certain elegance via the integration by parts mentioned in Appendix B, it led by the same route to the oversight described there.

Upon expansion, J provides 1 as its lead term and, as its next-order term, the sum $s_{1,1} + s_{2,2} + s_{3,3}$. This sum is taken to vanish in the linear theory of an incompressible medium, but it will not vanish even in the linear theory of a compressible medium nor in the nonlinear theory of an incompressible medium. In general, higher-order terms of the expansion must also be included.

With the retention of J, the integrand in (5.3) must be extended to include the product JJ' as a multiplier of LL'. The product produces LL' as its lead term, just as before, and the evaluation of correlations proceeds as in § 6 with respect to this term. The next-order term in JJ' permits the occurrence of correlations between the m mode of L, the n mode of L', and both the p mode of J and the q mode of J' separately, while higher-order terms include the p mode of J and the q mode of J' simultaneously, plus further combinations. These produce further correlations, as when $m = p$ and $n = q$, for example, and all these should be included in principle. All are implicitly included in the analysis of AJ89, but there is no evidence that they served any effective purpose there.

All of these complications could be included by extension of the analysis given here, but there is little motivation toward making the attempt. In particular, if the input Lagrangian spectrum contains only modes for which α_{m3} is substantially greater than the inverse of the scale height of density variation, the sum $s_{1,1}+s_{2,2}+s_{3,3}$ will continue to approximate to zero, and the first-order additional terms will effectively vanish (as AJ89 took them to do, by virtue of the polarization relations that were adopted). If this were not the case, the polarization relations that underlie the expressions given for I_1, I_2, and their counterparts would also need amendment. Further, if the higher-order additional terms were retained, they would be nonlinear in the Lagrangian spectrum and should in principle be complemented by other nonlinear terms that have been excluded here from the start.

Yet another complication arises. The transparency of the analysis here – most notably that of § 6 – has depended on a division of the input spectrum into $\hat{N}$ modes, with $\hat{N} \to$ infinity and $\hat{N}|\boldsymbol{S}_m|^2$ held constant. The nonlinear terms of J are such that, if admitted, they would become infinite with $\hat{N}$ (for a constant spectral distribution in $\boldsymbol{\alpha}$-space) and so would void the analysis entirely. This problem arises as much for the incompressible as for the compressible case, even though it is hidden in the former.

The main analysis can nevertheless be retained unchanged, if only the terms in J additional to 1 can be kept sufficiently small. This requires that $\hat{N}$ be restricted to some finite value, for which a corresponding maximum reduction of $|\boldsymbol{S}_m|$ occurs. The expansion of exponential factors given in § 6 is then valid only for a limited range of $\boldsymbol{k}$, not for the infinite range previously assumed. It can be shown that, for $Ri_L \gg 1$, this range extends far enough to provide the onset of the Eulerian spectral tail, but not far enough to permit the 'unwanted' leading term of the spectrum to come to exceed the 'wanted' second term, even if the 'unwanted' term is not removed via the arguments of Appendix C. The net result, both for the compressible and for the incompressible case, is the irrelevance of the 'unwanted' term but also an acknowledgment that the validity of the 'wanted' term has not been established to infinite k values. The analysis of Chunchuzov (2002) provides a more thorough, formal treatment of the limitation on k imposed by J.

The introduction of compressibility appears to have very little to recommend it and much arguing against it. The complexities of analysis that are found even in its absence, and the uncertainties that must attend any attempt to provide a truly representative spectrum for modelling purposes, suggest that compressibility is best avoided. This can be done at little cost so far as the applicability of the analysis is concerned, either in principle or, as in § 12, in application to observations.

Coriolis effects, if included (as by AJ89), would alter the analysis in three ways. First, they add a component of horizontal displacement transverse to the azimuth of propagation of each Lagrangian mode in turn, and this new component is in phase quadrature to the components of displacement that have been included here. The new component must be included both when the correlations of § 6 are effected and when the averaging in § 7 is conducted. In the former, it has the effect of introducing a combination of in-phase behaviour and phase-quadrature behaviour regardless of the nature of the field variable L. Secondly, Coriolis effects alter the polarization relations that have been employed in the body of the text, negating the utility of many of the spectral sums found in table 1 and requiring some revised and additional forms. Finally, Coriolis effects require that the new components of displacement and of velocity be taken into account when the Eulerian spectra of s_h and v_h are being determined.

These changes will alter the coefficients of the asymptotic terms that have been found here, but not the asymptotic forms themselves. A rough assessment of the effects suggests that whatever changes may occur will not be large. Then, too, the discussion in § 12 reveals that no large changes are needed to produce conformity with observation.

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